On polygonal relative equilibria in the N-vortex problem

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Abstract

Helmholtz's equations provide the motion of a system of N vortices which describes a planar incompressible fluid. A relative equilibrium is a particular solution of these equations for which the distances between the particles are invariant during the motion. In this article, we are interested in relative equilibria formed of concentric regular polygons of vortices. We show that in the case of one regular polygon with more than three vertices, a relative equilibrium requires equal vorticities. This result is the analogous of Perko-Walter-Elmabsout's in celestial mechanics. We also compute the number of relative equilibria with two concentric regular n-gons and the same vorticity on each n-gon, and we study the associated co-rotating points. This result completes previous studies for two regular n-gons when all the vortices have the same vorticity and when the total vorticity vanishes.

I Introduction

Let us consider a planar incompressible fluid with zero viscosity and zero velocity at infinity. Let us denote by $v(x,y) = (v_x(x,y), v_y(x,y)))$ the velocity field. According to the conservation of mass and the incompressibility (the density is constant and uniform), we have, at any time:

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0\cdot$$

Let us discretize it assuming that the curl of v (which in dimension 2 is a scalar) is a sum of Dirac delta functions based at z_1, ..., z_N. Then we have, at any point distinct from z_1, ... , z_N:

$$\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} = 0\cdot$$

These two equations express that, except at z_1, ... , z_N, the function $\bar{v}$ (conjugate of v) is holomorphic. According to Cauchy's formula, we obtain, for every $z_0 \neq z_1$, ... , z_N:

$$\bar{v}(z_0) = \frac{1}{2i\pi}\oint_{C_0} \frac{\bar{v}(z)}{z - z_0} dz\cdot$$

Let $C_1, \ldots, C_N$ be small contours around $z_1, \ldots, z_N$. As $\bar{v}$ is holomorphic except at the z_k and v vanishes at infinity, we have:

$$\bar{v}(z_0) = \frac{1}{2i\pi}\oint_{C_0} \frac{\bar{v}(z)}{z - z_0} dz = -\frac{1}{2i\pi}\sum_{k=1}^{N}\oint_{C_k} \frac{\bar{v}(z)}{z - z_0} dz\cdot$$

Each integral of the second member can be written as: $2\pi\Gamma_k/(z_j - z_0)$, where Γ_k is a real number. So we get:

$$v(z_0) = -\frac{i}{2\pi}\sum_{k=1}^{N}\frac{2\pi\Gamma_k}{\bar{z}_k - \bar{z}_0} = i\sum_{k=1}^{N}\frac{\Gamma_k}{\bar{z}_0 - \bar{z}_k}\cdot$$

So the fluid can be described as a set of N vortices whose positions $z_k(t)$ and *vorticities* $\Gamma_k(t)$ satisfy the following differential equations:

$$\dot{z}_k(t) = i\sum_{l\in\{1,\ldots,N\}\setminus\{k\}} \Gamma_l(t)\frac{z_k(t) - z_l(t)}{|z_k(t) - z_l(t)|^2} = i\sum_{l\in\{1,\ldots,N\}\setminus\{k\}} \frac{\Gamma_l(t)}{\bar{z}_k(t) - \bar{z}_l(t)}\cdot$$

Let us now write Navier-Stokes's equation with zero viscosity ($\ddot{z}$ denotes the particle derivative of the velocity):

$$\ddot{z} = \frac{Dv}{Dt} = -\nabla p\cdot$$

Due to the incompressibility condition (the divergence of v vanishes), and taking the curl of this equation, we can prove that the curl of v is constant. This condition is also known as Helmholtz's first theorem. It is equivalent to say that the Γ_k are constant. So the motion of the fluid is a solution of Helmholtz's equations (Helmholtz, 1858):

$$\dot{z}_k(t) = v_k(z_1(t), \ldots, z_N(t)) = i\sum_{l\in\{1,\ldots,N\}\setminus\{k\}} \Gamma_l\frac{z_k(t) - z_l(t)}{|z_k(t) - z_l(t)|^2} = i\sum_{l\in\{1,\ldots,N\}\setminus\{k\}} \frac{\Gamma_l}{\bar{z}_k(t) - \bar{z}_l(t)},$$

where the Γ_k are constant vorticities.

A good introduction to the problems and methods of vortex dynamics can be found in Aref (2007) and Newton (2001). In this paper, we are particularly interested in the relative equilibria motions. A motion of N vortices is said to be a *relative equilibrium* if, and only if, there exists a real number ω, called *angular velocity*, such that, for every k, l and for all time t:

$$z_l(t) - z_k(t) = e^{i\omega t}(z_l(0) - z_k(0))\cdot$$

Then one of the following statements is satisfied:

- If the z_k are constant, then the motion is said to be an *absolute equilibrium*. In this case, we have: $\omega = 0$.
- If there exists a *velocity of translation* $v \neq 0$ such that, for every k and for all time t: $z_k(t) = z_k(0) + tv$, then the motion is said to be a *rigid translation*. Again, we have: $\omega = 0$. We can notice that, for $\sum_{k=1}^{N}\Gamma_k \neq 0$, the center of vorticity:

$$\frac{1}{\sum_{k=1}^{N}\Gamma_k}\sum_{k=1}^{N}\Gamma_k z_k$$

is a constant of the motion, so there cannot be a rigid translation.

- If $\omega \neq 0$, then there exists a *center of rotation* Ω such that, for every k and for all time t:

$$z_k(t) = \Omega + e^{i\omega t}(z_k(0) - \Omega)\cdot$$

If, moreover: $\sum_{k=1}^{N} \Gamma_k \neq 0$, the center of rotation is the center of vorticity.

In fact, looking for these motions is equivalent to look for the configurations which generate them, as shown in the following proposition:

Proposition 1.1 *A motion of N vortices is a relative equilibrium if, and only if, at a certain time, there exists a real number ω such that we have, for every k, l: $v_l - v_k = i\omega(z_l - z_k)$. Then this condition is satisfied at any time, and ω is the angular velocity.*

- *It is an absolute equilibrium if, and only if, at a certain time, we have for every k: $v_k = 0$ (in this case: $\omega = 0$). Then this condition is satisfied at any time.*
- *It is a rigid translation if, and only if, at a certain time, there exists $v \neq 0$ such that we have, for every k: $v_k = v$ (again: $\omega = 0$). Then this condition is satisfied at any time, and v is the velocity of translation.*
- *It is a relative equilibrium with $\omega \neq 0$ if, and only if, at a certain time, there exists Ω such that we have, for every k: $v_k = i\omega(z_k - \Omega)$. Then this condition is satisfied at any time, and Ω is the center of the rotation.*

It is easy to check that when the N vortices have the same vorticity and they are located at the vertices of a regular N-gon, they form a relative equilibrium. We are interested in relative equilibria formed of p regular concentric polygons. Such configurations are also called *nested polygons*. Here, the polygons have the same number of vertices $n \geq 2$. Let $\rho = e^{i2\pi/n}$. We denote by $z_{k,d} = s_d\rho^k$, where $0 \leq k \leq n-1$ and $1 \leq d \leq p$, the positions of the n vertices of the d^{th} polygon. The s_d are non-vanishing complex numbers, every $s_{d'}/s_d$ is different from a ρ^K, where K is an integer. We denote by $v_{k,d}$ the velocities. For $p = 1$, we denote by Γ_k the vorticity of the k^{th} vertex and we assume $\Gamma_k \neq 0$. For $p \geq 2$, we assume that the vortices of a same polygon have the same non-vanishing vorticity and denote it by $\Gamma_d \neq 0$.

In this paper, we first prove that for $n \geq 4$ a regular polygon relative equilibrium ($p = 1$) only exists for equal vortices. This result is known in celestial mechanics (Perko and Walter, 1985; Elmabsout, 1988), where Helmholtz's equations have to be changed for Newton's equations:

$$\ddot{z}_k = \sum_{l \in \{1,\ldots,N\} \setminus \{k\}} \Gamma_l \frac{z_l - z_k}{|z_l - z_k|^3}\cdot$$

The Γ_k are then masses, and they have to be positive. In the N-vortex problem, we must take into account a new difficulty due to the possibility of having zero total vorticity. In this case, the center of vorticity is not defined, so we do not know the expression of a possible steady point.

Then we compute the relative equilibria of two concentric n-gons with the same vorticity on each n-gon. It is known that for these relative equilibria, each vortex is located on either a ray passing through the common center of the n-gons and a vertex of the other polygon or a ray passing through the common center of the n-gons and the middle point of a side of the other polygon (see Aref, Newton, Stremler, Tokieda and Vainchtein, 2002). These relative equilibria have already been computed for $\Gamma_1 = \Gamma_2$ and $\Gamma_1 = -\Gamma_2$ (see Aref, Newton, Stremler, Tokieda and Vainchtein, 2002) and in the analogous problem of celestial mechanics (the analogous of the vorticities are then positive), for the case where each vertex is located on a ray passing through the center and a vertex of the other polygon (Moeckel and Simó, 1995). They have also been computed for three concentric polygons with all equal vorticities (Aref and van Buren, 2005).

We also study the *co-rotating points* associated to these relative equilibria, i. e. the points which, when submitted to the only action of the relative equilibrium configuration, undergo the same rotation. This problem is equivalent to the restricted problem for relative equilibria in celestial mechanics, which consists in studying the possible relative equilibria for the configuration formed of the initial relative equilibrium and a particle with zero mass.

Adding a vortex at the center of the configuration, we can define analogous problems. In particular, a regular n-gon with equal vorticities remains a relative equilibrium when an additional vortex is added at the origin. Although we are not interested in these problems here, our arguments can be adapted to their study.

II Some useful identities

First, we have to express the angular velocity of a regular polygon.

Proposition 2.1 *For a relative equilibrium formed of a regular polygon with vertices $s\rho^k$, for $0 \leq k \leq n-1$, the angular velocity has the value:*

$$\omega = \frac{n-1}{2n|s|^2}\left(\sum_{k=0}^{n-1}\Gamma_k\right).$$

Proof. The proposition is evident for a rigid translation, as the angular velocity and the total vorticity vanish. Let us now suppose that the configuration is not rigidly translating. Then we have, for a certain Ω and for every $k \in \{0, \ldots, n-1\}$:

$$\omega(s\rho^k - \Omega) = \sum_{l \in \{0,\ldots,n-1\}\setminus\{k\}} \frac{\Gamma_l}{\bar{s}\rho^{-k} - \bar{s}\rho^{-l}}.$$

Dividing by ρ^k and taking the sum over k, we get:

$$ns\omega = \sum_{l=0}^{n-1} \sum_{k \in \{0,\ldots,n-1\}\setminus\{l\}} \frac{\Gamma_l}{\bar{s}(1-\rho^{k-l})},$$

according to the identity: $1+\rho+\rho^2+\ldots+\rho^{n-1}=0$.

$$n|s|^2\omega=\sum_{l=0}^{n-1}\sum_{k'=1}^{n-1}\frac{\Gamma_l}{1-\rho^{k'}}=\left(\sum_{l=0}^{n-1}\Gamma_l\right)\left(\sum_{k'=1}^{n-1}\frac{1}{1-\rho^{k'}}\right),$$

where $k'=k-l$

$$=\frac{1}{2}\left(\sum_{l=0}^{n-1}\Gamma_l\right)\left(\sum_{k=1}^{n-1}\frac{1}{1-\rho^k}+\sum_{k=1}^{n-1}\frac{1}{1-\rho^{-k}}\right)$$

as $\{\rho^{-k},1\le k\le n-1\}=\{\rho^k,1\le k\le n-1\}$

$$=\frac{1}{2}\left(\sum_{l=0}^{n-1}\Gamma_l\right)\sum_{k=1}^{n-1}\left(\frac{1}{1-\rho^k}+\frac{\rho^k}{\rho^k-1}\right)=\frac{n-1}{2}\left(\sum_{l=0}^{n-1}\Gamma_l\right)\cdot\ \text{QED.}$$

We will see in the next section that for $n\ge 4$, the Γ_k are equal indeed.

Now we express the velocity field generated by a regular polygon when the vorticities are equal.

Proposition 2.2 *The velocity field generated at the point z by n vortices on a regular polygon, with vorticity 1 and positions $s\rho^k$, has the following expression:*

$$v_s(z)=\frac{in\bar{z}^{n-1}}{\bar{z}^n-\bar{s}^n}=\frac{in\frac{z}{|z|^2}}{1-\left(\frac{\bar{s}}{\bar{z}}\right)^n}.$$

Proof. By definition, when $s=1$:

$$v_1(z)=i\sum_{k=0}^{n-1}\frac{z-\rho^k}{|z-\rho^k|^2}=i\sum_{k=0}^{n-1}\nabla(\ln|z-\rho^k|)=i\nabla\ln|P(z)|,$$

where $P(z)=\prod_{k=0}^{n-1}(z-\rho^k)$. The $z-\rho^k$ are coprime and divide z^n-1, so $P(z)$ divides z^n-1. Now $P(z)$ and z^n-1 have same degree and leading term. So $P(z)=z^n-1$. Hence: $v_1(z)=i\nabla\ln|z^n-1|$. Now, if we also assume $z\neq 0$, we have:

$$d\ln|z^n-1|=\left(\frac{z^n-1}{|z^n-1|^2}\,\Big|\,d(z^n-1)\right)=\left(\frac{1}{\bar{z}^n-1}\,\Big|\,nz^{n-1}dz\right)$$

$$=n|z|^{n-1}\left(\frac{1}{\bar{z}^n-1}\,\Big|\,\left(\frac{z}{|z|}\right)^{n-1}dz\right)=n|z|^{n-1}\left(\left(\frac{|z|}{z}\right)^{n-1}\frac{1}{\bar{z}^n-1}\,\Big|\,dz\right)$$

as the product by $(z/|z|)^{n-1}$ is an isometry. Hence:

$$v_1(z)=in|z|^{n-1}\frac{|z|^{n-1}}{z^{n-1}}\frac{1}{\bar{z}^n-1}=\frac{in\bar{z}^{n-1}}{\bar{z}^n-1}.$$

We can easily check that this formula also holds for $z=0$. When a system of vortices undergoes a similarity s, the velocities are multiplied by $1/\bar{s}$. Thus:

$$v_s(sz)=\frac{1}{\bar{s}}v_1(z),v_s(z)=\frac{1}{\bar{s}}v_1\left(\frac{z}{s}\right)=\frac{in\bar{z}^{n-1}}{\bar{z}^n-\bar{s}^n}\cdot\ \text{QED.}$$

The following proposition asserts that for a relative equilibrium formed of p concentric regular polygons, the center of the polygons does not move.

Proposition 2.3 *A relative equilibrium formed of concentric regular n-gons with the same vorticities on each n-gon is not a rigid translation. If $\omega \neq 0$, the center of rotation is the center of the polygons.*

Proof. We have to prove: $v_{k,d} = i\omega z_{k,d}$, for every polygon $d \in \{1, \ldots, p\}$ and every vertex $k \in \{0, \ldots, n-1\}$ of the polygon. Let $d \neq d_0 \in \{1, \ldots, p\}$. We have:

$$A_{d_0,d} = \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} \frac{z_{k,d_0} - z_{l,d}}{|z_{k,d_0} - z_{l,d}|^2} = \sum_{k=0}^{n-1} \rho^k \sum_{l=0}^{n-1} \frac{s_{d_0} - s_d \rho^{l-k}}{|s_{d_0} - s_d \rho^{l-k}|^2}$$

$$= \sum_{k=0}^{n-1} \rho^k \sum_{l=0}^{n-1} \frac{s_{d_0} - s_d \rho^l}{|s_{d_0} - s_d \rho^l|^2} = 0$$

as $\sum_{k=0}^{n-1} \rho^k = 0$. With the same argument, we can obtain:

$$B_{d_0} = \sum_{k=0}^{n-1} \sum_{l \in \{0,\ldots,n-1\} \setminus \{k\}} \frac{z_{k,d_0} - z_{l,d_0}}{|z_{k,d_0} - z_{l,d_0}|^2} = 0 \cdot$$

Thus for every $d_0 \in \{1, \ldots, p\}$, we have:

$$\sum_{k=0}^{n-1} v_{k,d_0} = \sum_{k=0}^{n-1} \left(\sum_{d \in \{1,\ldots,p\} \setminus \{d_0\}} \sum_{l=0}^{n-1} i\Gamma_d \frac{z_{k,d_0} - z_{l,d}}{|z_{k,d_0} - z_{l,d}|^2} + \sum_{l \in \{0,\ldots,n-1\} \setminus \{k\}} i\Gamma_{d_0} \frac{z_{k,d_0} - z_{l,d_0}}{|z_{k,d_0} - z_{l,d_0}|^2} \right)$$

$$= \sum_{d \in \{1,\ldots,p\} \setminus \{d_0\}} i\Gamma_d A_{d_0,d} + i\Gamma_{d_0} B_{d_0} = 0 \cdot$$

Hence, for every $k_0 \in \{0, \ldots, n-1\}$, $d_0 \in \{1, \ldots, p\}$:

$$v_{k_0,d_0} = v_{k_0,d_0} - \frac{1}{n} \sum_{k=0}^{n-1} v_{k,d_0} = \frac{1}{n} \sum_{k=0}^{n-1} (v_{k_0,d_0} - v_{k,d_0}) = \frac{i\omega}{n} \sum_{k=0}^{n-1} (z_{k_0,d_0} - z_{k,d_0}) = i\omega z_{k_0,d_0}$$

as $\sum_{k=0}^{n-1} z_{k,d_0} = 0$. QED.

III One polygon

It is known that a configuration formed of $n = 2$ or 3 vortices located at the vertices of a regular polygon is a relative equilibrium for any choice of the vorticities. In this section we are going to prove that the only relative equilibrium solutions for $n \geq 4$ vortices occur when the vorticities are all equal. The same result holds for the N-body problem (Perko and Walter, 1985; Elmabsout, 1988). We followed the ideas of Perko and Walter (1985), but we also had to consider additionally the zero total vorticity case where instead of rotation we have a translation of the system. This does not occur in celestial mechanics, where all the masses are positive.

Let us call *circulant matrix* a matrix of the form:

$$\begin{pmatrix} c_0 & c_1 & c_2 & \dots & c_{n-1} \\ c_{n-1} & c_0 & c_1 & \dots & c_{n-2} \\ \vdots & \vdots & \vdots & & \vdots \\ c_1 & c_2 & c_3 & \dots & c_0 \end{pmatrix}.$$

Let C be the circulant matrix with first row:

$$\left(\frac{n-1}{2n}, \frac{1}{1-\rho^{-1}} + \frac{n-1}{2n}\rho, \frac{1}{1-\rho^{-2}} + \frac{n-1}{2n}\rho^2, \dots, \frac{1}{1-\rho^{-(n-1)}} + \frac{n-1}{2n}\rho^{n-1}\right).$$

Let C_0 be the circulant matrix with first row:

$$\left(0, \frac{1}{1-\rho^{-1}}, \frac{1}{1-\rho^{-2}}, \dots, \frac{1}{1-\rho^{-(n-1)}}\right).$$

Lemma 3.1 *The matrices C and C_0 are diagonalizable. The eigenvectors of C and the corresponding eigenvalues are given by $\boldsymbol{v}_k = (1, \rho^k, \rho^{2k}, \dots, \rho^{(n-1)k})$ and $\lambda_k = (n-1)/2 - k$ for $0 \le k \le n-2$, $\lambda_{n-1} = 0$. The eigenvectors of C_0 and the corresponding eigenvalues are given by $\boldsymbol{v}_k = (1, \rho^k, \rho^{2k}, \dots, \rho^{(n-1)k})$ and $\lambda_k = (n-1)/2 - k$ for $0 \le k \le n-1$.*

Proof. According to the properties of circulant matrices (see for instance Marcus and Minc, 1992), the matrix C is diagonalizable with eigenvectors $\mathbf{v}_k = (1, \rho^k, \rho^{2k}, \dots, \rho^{(n-1)k})$. Moreover, we have:

$$\lambda_k = \sum_{l=0}^{n-1} c_l \rho^{kl} = \sum_{l=1}^{n-1} \frac{\rho^{kl}}{1-\rho^{-l}} + \frac{n-1}{2n} \sum_{l=0}^{n-1} \rho^{(k+1)l}$$

$$= \frac{1}{2}\left(\sum_{l=1}^{n-1} \frac{\rho^{kl}}{1-\rho^{-l}} + \sum_{l=1}^{n-1} \frac{\rho^{-kl}}{1-\rho^{l}}\right) + \frac{n-1}{2}\delta_{k,n-1}$$

as $\{\rho^{-l}, 1 \le l \le n-1\} = \{\rho^l, 1 \le l \le n-1\}$ ($\delta_{p,q}$ is the Kronecker delta)

$$= \frac{1}{2}\sum_{l=1}^{n-1}\left(\frac{\rho^{(k+1)l}}{\rho^l - 1} + \frac{\rho^{-kl}}{1-\rho^l}\right) + \frac{n-1}{2}\delta_{k,n-1} = \frac{1}{2}\sum_{l=1}^{n-1} \frac{\rho^{(k+1)l} - \rho^{-kl}}{\rho^l - 1} + \frac{n-1}{2}\delta_{k,n-1}$$

$$\frac{1}{2}\sum_{l=1}^{n-1} \frac{1}{\rho^l - 1} \sum_{m=-k}^{k} (\rho^{(m+1)l} - \rho^{ml}) + \frac{n-1}{2}\delta_{k,n-1} = \frac{1}{2}\sum_{m=-k}^{k}\sum_{l=1}^{n-1} \rho^{ml} + \frac{n-1}{2}\delta_{k,n-1}$$

$$= \frac{1}{2}\sum_{m=-k}^{k} f(m) + \frac{n-1}{2}\delta_{k,n-1} = \frac{n-1}{2} - k + \frac{n-1}{2}\delta_{k,n-1},$$

where $f(m) = -1$ if $m \neq 0$ and $f(0) = n-1$. The same reasoning holds for C_0, but we do not have the term:

$$\frac{n-1}{2n}\sum_{l=0}^{n-1} \rho^{(k+1)l} = \frac{n-1}{2}\delta_{k,n-1}$$

any more in the expression of λ_k. Thus we obtain:

$$\lambda_k = \frac{n-1}{2} - k$$

instead of:

$$\lambda_k = \frac{n-1}{2} - k + \frac{n-1}{2}\delta_{k,n-1} \cdot \text{ QED.}$$

Lemma 3.2 *If $a\boldsymbol{v}_{n-1} \in \mathbb{R}^n$ for some nonzero $a \in \mathbb{C}$, then $n = 2$. If n is odd and $a\boldsymbol{v}_{n-1} + b\boldsymbol{v}_{(n-1)/2} \in \mathbb{R}^n$ for some $a, b \in \mathbb{C}$ not simultaneously zero, then $n = 3$.*

Proof. From the form of $\mathbf{v}_{n-1} = (1, \rho^{-1}, \rho^{-2}, \ldots, \rho^{-(n-1)})$, we see that $a\mathbf{v}_{n-1}$ can not be real, except for $n = 2$ or $a = 0$.

For n odd, by writing $j = (n-1)/2$, we can rewrite more simply the vectors in terms of j as $\mathbf{v}_{n-1} = (1, \rho^{2j}, \rho^{4j}, \ldots, \rho^{2(n-1)j})$ and $\mathbf{v}_{(n-1)/2} = (1, \rho^{j}, \rho^{2j}, \ldots, \rho^{(n-1)j})$. We are going to prove that $a\mathbf{v}_{n-1} + b\mathbf{v}_{(n-1)/2} \in \mathbb{R}^n$ has only the trivial solution $a = b = 0$ if $j \geq 3$. This is equivalent to say that the scalar equations $-\bar{a}\bar{\rho}^{2kj} - \bar{b}\bar{\rho}^{kj} + a\rho^{2kj} + b\rho^{kj} = 0$ for $k = 0, \ldots,$ $n-1$ have only the complex solution $a = b = 0$ if $j \geq 2$. So we have a system of $n = 2j+1$ equations, one for each component. If $n \geq 5$ ($j \geq 2$), it is an overdetermined system in 4 unknowns. Let us consider in any case the subsystem consisting of the first 4 equations:

$$\begin{cases} -\bar{a} - \bar{b} + a + b = 0 \\ -\bar{a}\bar{\rho}^{2j} - \bar{b}\bar{\rho}^{j} + a\rho^{2j} + b\rho^{j} = 0 \\ -\bar{a}\bar{\rho}^{4j} - \bar{b}\bar{\rho}^{2j} + a\rho^{4j} + b\rho^{2j} = 0 \\ -\bar{a}\bar{\rho}^{6j} - \bar{b}\bar{\rho}^{3j} + a\rho^{6j} + b\rho^{3j} = 0 \end{cases}$$

Using the identity: $2j + 1 = n$, we can write the matrix of this system (in variables $-\bar{a}$, $-\bar{b}$, a, b) as:

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ \rho & \rho^{-j} & \rho^{-1} & \rho^{j} \\ \rho^2 & \rho^{-2j} & \rho^{-2} & \rho^{2j} \\ \rho^3 & \rho^{-3j} & \rho^{-3} & \rho^{3j} \end{pmatrix}.$$

We can notice that it is a Vandermonde matrix (see, for instance, Marcus and Minc, 1992). It is degenerate if, and only if, two terms of the second row are equal, which here cannot occur. QED.

Theorem 3.1 *Relative equilibrium solutions for a regular polygon with vorticities not all of them equal exist only in the cases $n = 2$ and $n = 3$. In particular, for a relative equilibrium formed of one regular polygon with $n \geq 4$, the total vorticity and the angular velocity do not vanish.*

Proof. We can suppose that the vortices have positions ρ^k. If the total vorticity does not vanish, we have, for every k:

$$i\omega(\rho^k - \Omega) = i \sum_{l \in \{0,\ldots,n-1\} \setminus \{k\}} \frac{\Gamma_l}{\rho^{-k} - \rho^{-l}},$$

where the center of rotation Ω is the center of vorticity:

$$\Omega = \frac{\sum_{l=0}^{n-1} \Gamma_l \rho^l}{\sum_{l=0}^{n-1} \Gamma_l}\cdot$$

This is equivalent to:

$$\sum_{l\in\{0,\ldots,n-1\}\setminus\{k\}} \frac{\Gamma_l}{\rho^{-k}-\rho^{-l}} + \omega\Omega = \omega\rho^k,$$

$$\sum_{l\in\{0,\ldots,n-1\}\setminus\{k\}} \frac{\Gamma_l}{\rho^{-k}-\rho^{-l}} + \omega\frac{\sum_{l=0}^{n-1} \Gamma_l \rho^l}{\sum_{l=0}^{n-1} \Gamma_l} = \omega\rho^k\cdot$$

Using the expression of ω given by Proposition 2.1:

$$\omega = \frac{n-1}{2n}\left(\sum_{l=0}^{n-1}\Gamma_l\right),$$

we get:

$$\sum_{l\in\{0,\ldots,n-1\}\setminus\{k\}} \frac{\Gamma_l}{\rho^{-k}-\rho^{-l}} + \frac{n-1}{2n}\sum_{l=0}^{n-1}\Gamma_l\rho^l = \rho^k\frac{n-1}{2n}\left(\sum_{l=0}^{n-1}\Gamma_l\right)\cdot$$

Dividing by ρ^k, we get:

$$\sum_{l\in\{0,\ldots,n-1\}\setminus\{k\}} \frac{\Gamma_l}{1-\rho^{k-l}} + \frac{n-1}{2n}\sum_{l=0}^{n-1}\Gamma_l\rho^{l-k} = \frac{n-1}{2n}\left(\sum_{l=0}^{n-1}\Gamma_l\right)\cdot$$

Thus we get the following system:

$$C\begin{pmatrix}\Gamma_0\\ \Gamma_1\\ \vdots\\ \Gamma_{n-1}\end{pmatrix} = \frac{n-1}{2n}\left(\sum_{l=0}^{n-1}\Gamma_l\right)\begin{pmatrix}1\\1\\ \vdots\\1\end{pmatrix} = \frac{1}{n}\left(\sum_{l=0}^{n-1}\Gamma_l\right)C\begin{pmatrix}1\\1\\ \vdots\\1\end{pmatrix}$$

according to Lemma 3.1. So $(\Gamma_0,\ldots,\Gamma_{n-1}) - (1/n)(\sum_{l=0}^{n-1}\Gamma_l)\mathbf{v}_0$ belongs to the kernel of C. According to Lemma 3.1, this nonzero vector, whose components are real, is collinear to $\mathbf{v}_{n-1}$ if n is even, and it is a linear combination of $\mathbf{v}_{n-1}$ and $\mathbf{v}_{(n-1)/2}$ if n is odd (they are the eigenvectors whose corresponding eigenvalue for C is zero). According to Lemma 3.2, this only occurs for $n = 2$ or 3.

If the total vorticity vanishes but the vorticities are otherwise arbitrary, we have, according to Proposition 2.1: $\omega = 0$. So there exists v such that we have, for every k:

$$\sum_{l\in\{0,\ldots,n-1\}\setminus\{k\}} \frac{\Gamma_l}{\rho^{-k}-\rho^{-l}} = v\cdot$$

This is equivalent to:

$$C_0 \begin{pmatrix} \Gamma_0 \\ \Gamma_1 \\ \vdots \\ \Gamma_{n-1} \end{pmatrix} = v \begin{pmatrix} 1 \\ \rho^{-1} \\ \vdots \\ \rho^{-(n-1)} \end{pmatrix} = C_0 \left(\frac{-2v}{n-1} \mathbf{v}_{n-1} \right),$$

according to Lemma 3.1. So $(\Gamma_0, \ldots, \Gamma_{n-1}) + (2v/(n-1))\mathbf{v}_{n-1}$ belongs to the kernel of C_0. This means that $(\Gamma_0, \ldots, \Gamma_{n-1})$ is collinear to $\mathbf{v}_{n-1}$ if n is even, and it is a linear combination of $\mathbf{v}_{n-1}$ and $\mathbf{v}_{(n-1)/2}$ if n is odd. According to Lemma 3.1 and Lemma 3.2 again, the solutions exist only for $n = 2$ or 3. QED.

Remark. If we assume from the beginning that the center of vorticity is the center of the polygon, this result can be proved in a simpler way. In this case, the motion is a *choreography*: the n vortices chase each other on the same curve with the same phase shift between two vortices. Now it can be shown that this cannot occur for distinct vorticities (Celli, 2003).

IV One polygon and one particle

Before studying relative equilibria with two polygons, let us list the co-rotating points of the relative equilibrium formed of a regular polygon with equal vorticities. We can easily compute them using Propositions 2.2 and 2.1 (see for instance Aref, Newton, Stremler, Tokieda and Vainchtein, 2002):

Theorem 4.1 *A regular n-gon with vortices at $s\rho^k$ and equal vorticities is a relative equilibrium with non-zero angular velocity. Let us consider the configuration formed of this polygon and a vortex with zero vorticity at position z. If the configuration is a relative equilibrium, then the argument of z/s is a multiple of π/n. Besides the relative equilibrium corresponding to the origin $z = 0$, there exists exactly one relative equilibrium with argument $2K\pi/n$, where K is an integer (the vortex with zero vorticity belongs to a ray containing the origin and a vertex of the polygon). If $n = 2$, there exists exactly one relative equilibrium with argument $2(K + 1/2)\pi/n$ (the vortex with zero vorticity belongs to a ray containing the origin and the middle point of two consecutive vertices of the polygon). If $n \geq 3$, there exist exactly two relative equilibria with argument $2(K + 1/2)\pi/n$.*

Let us notice that the number of relative equilibria is the same as the one obtained by Bang and Elmabsout (2004) in the analogous problem of celestial mechanics. However in that paper, the number of relative equilibria with argument $2(K + 1/2)\pi/n$ was obtained numerically.

V Two polygons

Let us consider a relative equilibrium formed of two concentric polygons, a n_1-gon and a n_2-gon, with the same vorticity Γ_1 or Γ_2 on each n-gon. According to Proposition 2.3, if

$n_1 = n_2$, the motion is not a rigid translation. Let us conversely suppose that the motion is not a rigid translation (this is always true when the total vorticity does not vanish). This is equivalent to say that at least one point does not move. By symmetry, it is equivalent to say that the center of the polygons does not move. Then, it can be shown that $n_1 = n_2$ (see Aref, Newton, Stremler, Tokieda and Vainchtein, 2002).

The study of the relative equilibria which satisfy these two equivalent hypotheses ($n_1 = n_2$ and no rigid translation) seems to go back to 1931 (see Havelock, 1931; Aref, Newton, Stremler, Tokieda and Vainchtein, 2002), with the particular case $\Gamma_1 = -\Gamma_2$. Later, the particular case $\Gamma_1 = \Gamma_2$ was also studied (see Aref, Newton, Stremler, Tokieda and Vainchtein, 2002). In this section, we compute the number of relative equilibria which satisfy these two hypotheses as a function of Γ_2/Γ_1.

Proposition 5.1 *Let us consider a configuration formed of two concentric regular n-polygons with the same vorticities on each n-gon. Let us denote by $s_1\rho^k$ the positions of the vortices of the first polygon and by $s_2\rho^k$ the positions of the vortices of the second polygon, and let us set: $x = |s_2/s_1|$. Then the configuration is a relative equilibrium if, and only if, one of the two following conditions is satisfied:*

- *The ratio s_2/s_1 has argument $2K\pi/n$, where K is an integer (any vortex of the second polygon belongs to a ray containing the origin and a vertex of the first polygon), and the following identity is satisfied:*

$$\left(x^2 - \left(\frac{\Gamma_2}{\Gamma_1} + \frac{2n}{n-1}\right)\right)\left(x^n - \left(\frac{2n}{n-1}\frac{\Gamma_2}{\Gamma_1} + 1\right)\right) = \frac{2n}{n-1}\left(\left(\frac{\Gamma_2}{\Gamma_1}\right)^2 + \frac{2n}{n-1}\frac{\Gamma_2}{\Gamma_1} + 1\right) \tag{1}$$

- *The ratio s_2/s_1 has argument $2(K+1/2)\pi/n$, where K is an integer (any vortex of the second polygon belongs to a ray containing the origin and the middle point of two consecutive vertices of the first polygon), and the following identity is satisfied:*

$$\left(x^2 - \left(\frac{\Gamma_2}{\Gamma_1} + \frac{2n}{n-1}\right)\right)\left(x^n + \left(\frac{2n}{n-1}\frac{\Gamma_2}{\Gamma_1} + 1\right)\right) = -\frac{2n}{n-1}\left(\left(\frac{\Gamma_2}{\Gamma_1}\right)^2 + \frac{2n}{n-1}\frac{\Gamma_2}{\Gamma_1} + 1\right) \tag{2}$$

Proof. For any k, using Propositions 2.2 and 2.1, we have:

$$\frac{v_{k,1}}{z_{k,1}} = \Gamma_2\frac{v_{s_2}(z_{k,1})}{z_{k,1}} + \Gamma_1\frac{i(n-1)}{2|z_{k,1}|^2} = \frac{in\Gamma_2}{|s_1|^2\left(1 - \left(\frac{\bar{s}_2}{\bar{s}_1}\right)^n\right)} + \frac{i(n-1)\Gamma_1}{2|s_1|^2}\cdot$$

So, for any k, $l \in \{0, \ldots, n-1\}$, we have:

$$\frac{v_{k,1}}{z_{k,1}} = \frac{v_{l,1}}{z_{l,1}}, \quad \frac{v_{k,2}}{z_{k,2}} = \frac{v_{l,2}}{z_{l,2}}\cdot$$

So, according to Proposition 2.3, the configuration is a relative equilibrium if, and only if:

$$\frac{v_{0,1}}{z_{0,1}} = \frac{v_{0,2}}{z_{0,2}} \in i\mathbb{R}\cdot$$

From the expression of the $v_{k,1}/z_{k,1}$ that we have just computed, the proposition $v_{0,1}/z_{0,1} \in i\mathbb{R}$ implies that $(s_2/s_1)^n \in \mathbb{R}$. Thus the argument of s_2/s_1 is a multiple of π/n. If s_2/s_1 has argument $2K\pi/n$, the identity $v_{0,1}/z_{0,1} = v_{0,2}/z_{0,2}$ is equivalent to:

$$\frac{n\Gamma_2 x^2}{1-x^n} + \frac{(n-1)\Gamma_1 x^2}{2} = \frac{n\Gamma_1 x^n}{x^n - 1} + \frac{(n-1)\Gamma_2}{2},$$

$$\frac{(n-1)\Gamma_1}{2}x^{n+2} - \left(n\Gamma_1 + \frac{(n-1)\Gamma_2}{2}\right)x^n - \left(n\Gamma_2 + \frac{(n-1)\Gamma_1}{2}\right)x^2 + \frac{(n-1)\Gamma_2}{2} = 0,$$

which is equivalent to the first condition. If s_2/s_1 has argument $2(K+1/2)\pi/n$, we can check that the identity $v_{0,1}/z_{0,1} = v_{0,2}/z_{0,2}$ is equivalent to the second condition. QED.

In order to solve these equations, we will need two technical results. In any case we need to find the real roots of polynomials of the form:

$$G_n(\alpha, \beta, \gamma, x) = (x^2 - \alpha)(x^n - \beta) - \gamma\cdot$$

In terms of the new variable $t = x^2$, they become:

$$H_n(\alpha, \beta, \gamma, t) = (t - \alpha)(t^{n/2} - \beta) - \gamma,$$

whose derivative with respect to t is $F_n = \partial H_n/\partial t$.

We set:

$$\mathbb{R}_+ = \{x \in \mathbb{R}, x \geq 0\}; \mathbb{R}_+^* = \{x \in \mathbb{R}, x > 0\}; \mathbb{R}_- = \{x \in \mathbb{R}, x \leq 0\}; \mathbb{R}_-^* = \{x \in \mathbb{R}, x < 0\}\cdot$$

Lemma 5.1 *For any real numbers α, β, $t > 0$, and for any natural number $n \geq 2$, we define:*

$$F_n(\alpha, \beta, t) = \left(\frac{n}{2} + 1\right)t^{n/2} - \alpha\frac{n}{2}t^{n/2-1} - \beta\cdot$$

Then the following results hold:

- *For every α, β such that $\alpha+\beta > 0$, the equation $F_2(\alpha, \beta, t) = 0$ has exactly one solution $t > 0$, whose expression is $(\alpha+\beta)/2$.*
- *For every α, β such that $\alpha + \beta \leq 0$, the equation $F_2(\alpha, \beta, t) = 0$ has no solution $t > 0$.*

For $n \geq 3$ we define $\Delta_1 = \{(\alpha, 0), \alpha < 0\}$, $\Delta_2^n = \{(\alpha, \beta) \in \mathbb{R}_+^* \times \mathbb{R}_-^*, \beta^2 = \alpha^n(n-2)^{n-2}/(n+2)^{n-2}\}$, $\Delta_3 = \{(\alpha, 0), \alpha > 0\}$. Let $\mathcal{D}_{12}^n$, $\mathcal{D}_{23}^n$, $\mathcal{D}_{31}$ be the open domains in the plane which are bounded respectively by (Δ_1, Δ_2^n), (Δ_2^n, Δ_3), (Δ_3, Δ_1). Then we have:

- *For every $(\alpha, \beta) \in \Delta_1$, the equation $F_n(\alpha, \beta, t) = 0$ has no solution $t > 0$.*
- *For every $(\alpha, \beta) \in \Delta_2^n$, the equation $F_n(\alpha, \beta, t) = 0$ has exactly one solution $t > 0$, whose expression is $\alpha(n-2)/(n+2)$.*
- *For every $(\alpha, \beta) \in \Delta_3$, the equation $F_n(\alpha, \beta, t) = 0$ has exactly one solution $t > 0$, whose expression is $\alpha n/(n+2)$.*

- *For every $(\alpha, \beta) \in \mathcal{D}_{12}^n$, the equation $F_n(\alpha, \beta, t) = 0$ has no solution $t > 0$.*

- *There exist two analytic functions φ_1^n and φ_2^n: $\mathcal{D}_{23}^n \to \mathbb{R}_+^*$ such that, for every $(\alpha, \beta) \in \mathcal{D}_{23}^n$, the equation $F_n(\alpha, \beta, t) = 0$ has exactly two solutions $t > 0$, whose expressions are $\varphi_1^n(\alpha, \beta)$ and $\varphi_2^n(\alpha, \beta)$.*

- *There exists an analytic function φ^n: $\mathcal{D}_{31} \to \mathbb{R}_+^*$ such that, for every $(\alpha, \beta) \in \mathcal{D}_{31}$, the equation $F_n(\alpha, \beta, t) = 0$ has exactly one solution $t > 0$, whose expression is $\varphi^n(\alpha, \beta)$.*

In any case, for every fixed (α, β), unless $(\alpha, \beta) \in \Delta_2^n$, the function $t \to F_n(\alpha, \beta, t)$, that we will also denote by $F_n(\alpha, \beta, .)$, changes sign at its strictly positive zeros.

Proof. We have to study the sign of $\partial F_n/\partial t$. As F_n and $\partial F_n/\partial t$ do not vanish simultaneously on $\mathcal{D}_{23}^n \times \mathbb{R}_+^*$ and $\mathcal{D}_{31} \times \mathbb{R}_+^*$, the implicit function theorem implies the analyticity of φ_1^n, φ_2^n and φ^n. QED.

Lemma 5.2 *For any real numbers α, β, γ, $x > 0$, and for any natural number $n \geq 2$, we define:*

$$G_n(\alpha, \beta, \gamma, x) = (x^2 - \alpha)(x^n - \beta) - \gamma\cdot$$

Then the following results hold:

- *For every α, β, γ such that $\gamma < -(\beta - \alpha)^2/4$, the equation $G_2(\alpha, \beta, \gamma, x) = 0$ has no solution $x > 0$.*

- *For every α, β, γ such that $\gamma = -(\beta - \alpha)^2/4$ and $\alpha + \beta \leq 0$, the equation $G_2(\alpha, \beta, \gamma, x) = 0$ has no solution $x > 0$.*

- *For every α, β, γ such that $\gamma = -(\beta - \alpha)^2/4$ and $\alpha + \beta > 0$, the equation $G_2(\alpha, \beta, \gamma, x) = 0$ has exactly one solution $x > 0$, whose expression is $(\alpha + \beta)/2$.*

- *For every α, β, γ such that $-(\beta - \alpha)^2/4 < \gamma \leq \alpha\beta$ and $\alpha + \beta < 0$ (let us notice that we have, for every α, β: $\alpha\beta \geq -(\beta - \alpha)^2/4$), the equation $G_2(\alpha, \beta, \gamma, x) = 0$ has no solution $x > 0$.*

- *For every α, β, γ such that $\alpha + \beta = 0$ or $\gamma > \alpha\beta (\geq -(\beta - \alpha)^2/4)$ or ($\gamma = \alpha\beta > -(\beta - \alpha)^2/4$ and $\alpha + \beta > 0$), the equation $G_2(\alpha, \beta, \gamma, x) = 0$ has exactly one solution $x > 0$, whose expression is $(\alpha + \beta + \sqrt{(\beta - \alpha)^2 + 4\gamma})/2$.*

- *For every α, β, γ such that $-(\beta - \alpha)^2/4 < \gamma < \alpha\beta$ and $\alpha + \beta > 0$, the equation $G_2(\alpha, \beta, \gamma, x) = 0$ has exactly two solutions $x > 0$, whose expressions are $(\alpha + \beta - \sqrt{(\beta - \alpha)^2 + 4\gamma})/2$ and $(\alpha + \beta + \sqrt{(\beta - \alpha)^2 + 4\gamma})/2$.*

Let $n \geq 3$. We have:

- *For every α, β, the equation $G_n(\alpha, \beta, 0, x) = 0$ has exactly 0, 1 or 2 solutions $x > 0$, according to the sign of α and β.*

- *For every α, β, γ such that $\alpha\beta \geq \gamma > 0$ and α or $\beta \leq 0$, the equation $G_n(\alpha, \beta, \gamma, x) = 0$ has no solution $x > 0$.*

- *For every $\gamma > 0$, α, β such that $\alpha\beta < \gamma$ or ($\alpha\beta = \gamma$ and α and $\beta > 0$), the equation $G_n(\alpha, \beta, \gamma, x) = 0$ has exactly one solution $x > 0$.*

- *For every α, β, γ such that $\alpha\beta > \gamma > 0$ and α and $\beta > 0$, the equation $G_n(\alpha, \beta, \gamma, x) = 0$ has exactly two solutions $x > 0$.*

- *For every α, β, γ such that $\alpha\beta < \gamma < 0$ and $(\alpha, \beta) \in \Delta_2^n \cup \mathcal{D}_{12}^n \cup \mathcal{D}_{31}$ (union of sets defined in Lemma 5.1, the equation $G_n(\alpha, \beta, \gamma, x) = 0$ has exactly one solution $x > 0$.*

- *For every $\gamma < 0$, α, β such that $\alpha\beta \geq \gamma$ and $(\alpha, \beta) \in \Delta_1 \cup \Delta_2^n \cup \mathcal{D}_{12}^n$, the equation $G_n(\alpha, \beta, \gamma) = 0$ has no solution $x > 0$.*

- *In the other cases with $\gamma < 0$, the equation $G_n(\alpha, \beta, \gamma, x) = 0$ can have between zero and three solutions, depending on the sign of $\alpha\beta - \gamma$, $G_n(\alpha, \beta, \gamma, \varphi^n(\alpha, \beta))$, $G_n(\alpha, \beta, \gamma, \varphi_1^n(\alpha, \beta))$, $G_n(\alpha, \beta, \gamma, \varphi_2^n(\alpha, \beta))$, $G_n(\alpha, \beta, \gamma, \alpha n/(n+2))$, where φ^n, φ_1^n and φ_2^n are defined in Lemma 5.1.*

Proof. For $n = 2$ the results hold easily since the equation $G_2(\alpha, \beta, \gamma, x) = 0$ is biquadratic in x. For the rest of the proof we assume $n \geq 3$.

Let us consider the case: $\alpha\beta \geq \gamma > 0$ and α or $\beta \leq 0$. We necessarily have: α and $\beta < 0$. So $G_n(\alpha, \beta, \gamma, .)$ is strictly increasing. As its limit at 0 is $\alpha\beta - \gamma \geq 0$, it never vanishes.

Let us consider the case: $\gamma > 0$ and ($\gamma > \alpha\beta$ or ($\gamma = \alpha\beta$ and α and $\beta > 0$)). For every solution $x > 0$ of the equation $(x^2 - \alpha)(x^n - \beta) = \gamma > 0$, the factors $x^2 - \alpha$ and $x^n - \beta$ have the same sign. Thus: ($x^2 < \alpha$ and $x^n < \beta$) or ($x^2 > \alpha$ and $x^n > \beta$). On the interval $\{x \in \mathbb{R}_+^*, x^2 < \alpha \text{ and } x^n < \beta\}$ (which is possibly empty), the function $G_n(\alpha, \beta, \gamma, .)$ is strictly decreasing, and its limit at 0 is $\alpha\beta - \gamma \leq 0$. So in fact, a solution $x > 0$ satisfies: $x^2 > \alpha$ and $x^n > \beta$.

Let us suppose α and $\beta \leq 0$. We have: $\{x \in \mathbb{R}_+^*, x^2 > \alpha \text{ and } x^n > \beta\} = \mathbb{R}_+^*$. The function $G_n(\alpha, \beta, \gamma, .)$ is strictly increasing, its limit at 0 is $\alpha\beta - \gamma < 0$ and its limit at $+\infty$ is $+\infty$. So it vanishes exactly once.

Let us suppose α or $\beta > 0$. We have: $\{x \in \mathbb{R}_+^*, x^2 > \alpha \text{ and } x^n > \beta\} =]\sqrt{\alpha}, +\infty[$ or $]\beta^{1/n}, +\infty[$. The function $G_n(\alpha, \beta, \gamma, .)$ is strictly increasing on this interval, its limit at $\sqrt{\alpha}$ or $\beta^{1/n}$ is $-\gamma < 0$ and its limit at $+\infty$ is $+\infty$. So it vanishes exactly once.

Let us consider the case: $\alpha\beta > \gamma > 0$ and α and $\beta > 0$. A solution $x > 0$ of $(x^2 - \alpha)(x^n - \beta) = \gamma > 0$ belongs to $]0, \min(\sqrt{\alpha}, \beta^{1/n})[$ or to $]\max(\sqrt{\alpha}, \beta^{1/n}), +\infty[$. The function $G_n(\alpha, \beta, \gamma, .)$ decreases strictly from $\alpha\beta - \gamma > 0$ to $-\gamma < 0$ on the first interval, and increases strictly from $-\gamma < 0$ to $+\infty$ on the second. So it vanishes exactly twice.

Let us consider the case: $\alpha\beta < \gamma < 0$ and $(\alpha, \beta) \in \Delta_1 \cup \Delta_2^n \cup \Delta_3 \cup \mathcal{D}_{12}^n \cup \mathcal{D}_{31}$. Let us set: $H_n(\alpha, \beta, \gamma, x^2 = t) = G_n(\alpha, \beta, \gamma, x)$. We have: $\partial H_n / \partial t = F_n$, where F_n is defined in Lemma 5.1. Thus: $\partial G_n / \partial x(\alpha, \beta, \gamma, x)$ has the same sign as $F_n(\alpha, \beta, x^2)$. According to Lemma 5.1, when $(\alpha, \beta) \in \Delta_1 \cup \Delta_2^n \cup \Delta_3 \cup \mathcal{D}_{12}^n \cup \mathcal{D}_{31}$, the function $F_n(\alpha, \beta, .)$ is strictly positive, or strictly negative then strictly positive, or strictly positive except at one point. Thus, the function $G_n(\alpha, \beta, \gamma, .)$ is strictly increasing, or strictly decreasing then strictly increasing. As its limit at 0 is $\alpha\beta - \gamma < 0$ and its limit at $+\infty$ is $+\infty$, it vanishes exactly once.

We deal with the other cases with $\gamma < 0$ in an analogous way. In order to deal with the case $\alpha\beta \geq \gamma$ and $(\alpha, \beta) \in \Delta_1 \cup \Delta_2^n \cup \mathcal{D}_{12}^n$, we notice that $F_n(\alpha, \beta, .)$ is strictly positive

(except possibly at one point). QED.

Remark. We can also obtain the number of solutions in the cases: $\gamma > 0$ and in the case: $\alpha\beta < \gamma < 0$ and $(\alpha, \beta) \in \mathcal{D}_{31}$ with a simple graphical reasoning. Let (H) be the hyperbola with equation: $XY = \gamma$. Let $(\Delta) = \{(X, Y) \in \mathbb{R}^2, X > -\alpha, Y > -\beta, (X + \alpha)^n = (Y + \beta)^2\}$. Setting $x = \sqrt{X + \alpha} = (Y + \beta)^{1/n}$, we can notice that the number of solutions of the equation $G_n(\alpha, \beta, \gamma, x) = 0$ is equal to the number of intersections of (H) and (Δ).

If $\alpha\beta \geq \gamma > 0$ and α or $\beta \leq 0$, then α and $\beta < 0$, and (Δ) is located at the top right of the two connected components of (H): there is no solution.

If $\gamma > 0$ and ($\gamma > \alpha\beta$ or ($\gamma = \alpha\beta$ and α and $\beta > 0$)), the origin $(-\alpha, -\beta)$ of (Δ) is located between the two connected components of (H), possibly in the component at the bottom left: there is exactly one solution.

If $\alpha\beta > \gamma > 0$ and α and $\beta > 0$, the origin of (Δ) is located at the bottom left of the two connected components of (H): there are exactly two solutions.

If $\alpha\beta < \gamma < 0$ and $\beta > 0$ (which implies $\alpha < 0$), the origin of (Δ) is located at the bottom right of the two connected components of (H): there is exactly one solution.

Now we can compute the number of relative equilibria formed of two concentric regular n-gons for a given vorticity on each polygon.

Theorem 5.1 *Let us consider a configuration formed of two concentric regular n-gons with the same vorticities on each n-gon. We have seen in Proposition 5.1 that if the configuration is a relative equilibrium, then the argument of s_2/s_1 is a multiple of π/n.*

- *Let us suppose that $n = 2$ and Γ_1 and Γ_2 have the same sign. Then there exist exactly two relative equilibria with argument $2K\pi/n$. And there exists exactly one relative equilibrium with argument $2(K + 1/2)\pi/n$.*

- *Let us suppose that $n = 2$ and Γ_1 and Γ_2 have opposite signs, with $\Gamma_1 + \Gamma_2 \neq 0$. Then there exists exactly one relative equilibrium with argument $2K\pi/n$. And there exist exactly two relative equilibria with argument $2(K + 1/2)\pi/n$.*

- *Let us suppose that $n \geq 3$ and Γ_1 and Γ_2 have the same sign. Then there exist exactly two relative equilibria with argument $2K\pi/n$. And there exist exactly one, two or three relative equilibria with argument $2(K + 1/2)\pi/n$.*

For every integer $n \geq 2$ we define:

$$\mu_n = \frac{n}{n-1} + \sqrt{\left(\frac{n}{n-1}\right)^2 - 1}.$$

- *Let us suppose that $n \geq 3$ and $-1/\mu_n < \Gamma_2/\Gamma_1 < 0$. Then there exists exactly one relative equilibrium with argument $2K\pi/n$. And there exist exactly zero or one or two relative equilibria with argument $2(K + 1/2)\pi/n$.*

- *Let us suppose that $n \geq 3$ and $\Gamma_2/\Gamma_1 = -1/\mu_n$ or $-\mu_n$. Then there exists exactly one relative equilibrium with argument $2K\pi/n$. And there exist exactly two relative equilibria with argument $2(K + 1/2)\pi/n$.*

- *Let us suppose that $n \geq 3$ and $-\mu_n < \Gamma_2/\Gamma_1 < -1/\mu_n$, with $\Gamma_2/\Gamma_1 \neq -1$. Then there exist exactly one or two or three relative equilibria with argument $2K\pi/n$. And there exist exactly two relative equilibria with argument $2(K+1/2)\pi/n$.*

- *Let us suppose that $n \geq 3$ and $\Gamma_2/\Gamma_1 < -\mu_n$. Then there exists exactly one relative equilibrium with argument $2K\pi/n$. And there exist exactly zero or one or two relative equilibria with argument $2(K+1/2)\pi/n$.*

- *Let us suppose that $\Gamma_1+\Gamma_2 = 0$. Then the configuration cannot be a relative equilibrium with argument $2K\pi/n$ for any value of x. And there exist exactly two relative equilibria with argument $2(K+1/2)\pi/n$.*

Proof. According to Proposition 5.1, the number of relative equilibria with argument $2K\pi/n$ is equal to the number of solutions of equation (1) in $\mathbb{R}_+ \setminus \{0,1\}$. This equation has 1 as solution if, and only if: $\Gamma_2/\Gamma_1 = -1$. Thus, the number of relative equilibria with argument $2K\pi/n$ is equal to the number of solutions of equation (1) in $\mathbb{R}_+^*$ if $\Gamma_1+\Gamma_2 \neq 0$, and it is equal to the number of solutions of equation (1) in $\mathbb{R}_+^*$, deleting solution 1, if $\Gamma_1+\Gamma_2 = 0$. According to Proposition 5.1, the number of relative equilibria with argument $2(K+1/2)\pi/n$ is equal to the number of solutions of equation (2) in $\mathbb{R}_+^*$. So we have to look for the strictly positive solutions of equations (1) and (2). When $\Gamma_1+\Gamma_2 = 0$, we have to remove the solution 1 of equation (1).

Equations (1) and (2) have the form: $G_n(\alpha,\beta,\gamma,x) = 0$, where G_n is defined in Lemma 5.2. If $n = 2$, we can notice that we have, for equation (1):

$$\alpha+\beta > 0 \Leftrightarrow \frac{\Gamma_2}{\Gamma_1} > -1; \quad \gamma > -\frac{(\beta-\alpha)^2}{4}; \quad \alpha\beta > \gamma \Leftrightarrow \frac{\Gamma_2}{\Gamma_1} > 0\cdot$$

And we have, for equation (2):

$$\alpha+\beta > 0 \Leftrightarrow \frac{\Gamma_2}{\Gamma_1} < 1; \quad \gamma > -\frac{(\beta-\alpha)^2}{4}; \quad \alpha\beta > \gamma \Leftrightarrow \frac{\Gamma_2}{\Gamma_1} < 0\cdot$$

Thus, when $\Gamma_2/\Gamma_1 > 0$, we have, for equation (1): $\alpha+\beta > 0$, $\alpha\beta > \gamma$. According to Lemma 5.2, there are exactly two solutions. And we have, for equation (2): $\alpha\beta < \gamma$. So there is exactly one solution. When $\Gamma_2/\Gamma_1 < 0$, we have, for equation (1): $\alpha\beta < \gamma$. There is exactly one solution. This solution is different from 1 if, and only if: $\Gamma_2/\Gamma_1 \neq -1$. And we have, for equation (2): $\alpha+\beta > 0$, $\alpha\beta > \gamma$. There are exactly two solutions.

For every $n \geq 3$, let us set: $\lambda_n = 2n/(n-1)$. We have: $\lambda_n > \mu_n$. Then, for equation (1), we have:

$$\alpha > 0 \Leftrightarrow \frac{\Gamma_2}{\Gamma_1} > -\lambda_n; \quad \beta > 0 \Leftrightarrow \frac{\Gamma_2}{\Gamma_1} > -\frac{1}{\lambda_n}; \quad \gamma > 0 \Leftrightarrow \frac{\Gamma_2}{\Gamma_1} < -\mu_n \text{ or } \frac{\Gamma_2}{\Gamma_1} > -\frac{1}{\mu_n}; \quad \alpha\beta > \gamma \Leftrightarrow \frac{\Gamma_2}{\Gamma_1} > 0\cdot$$

For equation (2), we have:

$$\alpha > 0 \Leftrightarrow \frac{\Gamma_2}{\Gamma_1} > -\lambda_n; \quad \beta > 0 \Leftrightarrow \frac{\Gamma_2}{\Gamma_1} < -\frac{1}{\lambda_n}; \quad \gamma > 0 \Leftrightarrow -\mu_n < \frac{\Gamma_2}{\Gamma_1} < -\frac{1}{\mu_n}; \quad \alpha\beta > \gamma \Leftrightarrow \frac{\Gamma_2}{\Gamma_1} < 0\cdot$$

Thus, when $\Gamma_2/\Gamma_1 > 0$, we have, for equation (1): $\alpha > 0$, $\beta > 0$, $\gamma > 0$, $\alpha\beta > \gamma$. So there are exactly two solutions. And we have, for equation (2): $\alpha > 0$, $\beta < 0$, $\gamma < 0$, $\alpha\beta < \gamma$. So $(\alpha, \beta) \in \Delta_2^n \cup \mathcal{D}_{12}^n \cup \mathcal{D}_{23}^n$. There are exactly one or two or three solutions, depending on the sign of $\beta^2 - \alpha^n(n-2)^{n-2}/(n+2)^{n-2}$ and, eventually, on the sign of $G_n(\alpha, \beta, \gamma, \varphi_1^n(\alpha, \beta))$ and $G_n(\alpha, \beta, \gamma, \varphi_2^n(\alpha, \beta))$. We deal with the other cases with $n \geq 3$ in the same way. QED.

Remark. The number of relative equilibria with argument $2K\pi/n$ in the particular case where the vorticities have the same sign is the same as in the analogous problem of celestial mechanics (the vorticities are then masses, which are always positive), solved by Moeckel and Simó (1995).

Theorem 5.2 *Let us consider a configuration formed of two concentric regular n-gons with the same vorticities on each n-gon. If the configuration is an absolute equilibrium (which, according to Proposition 2.3, is equivalent to a relative equilibrium with zero velocity), we have: $\Gamma_2/\Gamma_1 = -\mu_n$ or $-1/\mu_n$, where μ_n is defined in Theorem 5.1. For each of these two values of Γ_2/Γ_1 (which, in fact, define the same equilibria, as each one is the inverse of the other: we just have to permute the indices 1 and 2 of the polygons), only one of the three relative equilibria mentioned in Theorem 5.1 is an absolute equilibrium. It is one of the two relative equilibria with argument $2(K+1/2)\pi/n$. The external polygon is the polygon whose vorticity has larger absolute value.*

Proof. As in the proof of Proposition 5.1, we have, for any k, $l \in \{0, \ldots, n-1\}$, using again Propositions 2.2 and 2.1:

$$\frac{v_{k,1}}{z_{k,1}} = \frac{v_{l,1}}{z_{l,1}}, \; \frac{v_{k,2}}{z_{k,2}} = \frac{v_{l,2}}{z_{l,2}}\cdot$$

So the configuration is an absolute equilibrium if, and only if: $v_{0,1} = v_{0,2} = 0$. According to Propositions 2.2 and 2.1, this is equivalent to:

$$\frac{\Gamma_2 n}{1 - \left(\frac{\bar{s}_2}{\bar{s}_1}\right)^n} + \frac{\Gamma_1(n-1)}{2} = \frac{\Gamma_1 n}{1 - \left(\frac{\bar{s}_1}{\bar{s}_2}\right)^n} + \frac{\Gamma_2(n-1)}{2} = 0\cdot$$

This is also equivalent to:

$$\left(\frac{s_2}{s_1}\right)^n = 1 + \frac{2n}{n-1}\frac{\Gamma_2}{\Gamma_1} = \frac{1}{1 + \frac{2n}{n-1}\frac{\Gamma_1}{\Gamma_2}}\cdot$$

The identity of the two last members is equivalent to:

$$\frac{2n}{n-1}\frac{\Gamma_2}{\Gamma_1} + \frac{2n}{n-1}\frac{\Gamma_1}{\Gamma_2} + \left(\frac{2n}{n-1}\right)^2 = 0,$$

$$\left(\frac{\Gamma_2}{\Gamma_1}\right)^2 + \frac{2n}{n-1}\frac{\Gamma_2}{\Gamma_1} + 1 = 0,$$

$$1 + \frac{2n}{n-1}\frac{\Gamma_2}{\Gamma_1} = -\left(\frac{\Gamma_2}{\Gamma_1}\right)^2\cdot$$

So the configuration is an absolute equilibrium if, and only if:

$$\left(\frac{s_2}{s_1}\right)^n = -\left(\frac{\Gamma_2}{\Gamma_1}\right)^2 = 1 + \frac{2n}{n-1}\frac{\Gamma_2}{\Gamma_1}\cdot$$

We obtain the value of Γ_2/Γ_1 solving the second degree equation which expresses the identity of the two last members. The identity of the two first members and the negativity of $-(\Gamma_2/\Gamma_1)^2$ give us the argument and the modulus of s_2/s_1. QED.

Remark. For any absolute equilibrium of N vortices with positions $z_1, \ldots, z_N$ and vorticities $\Gamma_1, \ldots, \Gamma_N$, we have:

$$\sum_{1\leq k<l\leq N} \Gamma_k\Gamma_l = 0$$

(see for instance O'Neil, 1987, for more details). In the more general problem where the velocities have terms $(z_k - z_l)/||z_k - z_l||^\alpha$, this identity would have the form:

$$\sum_{1\leq k<l\leq N} \frac{\Gamma_k\Gamma_l}{||z_k - z_l||^{\alpha-2}} = 0\cdot$$

Only in the present case with $\alpha = 2$, this equation does not involve the mutual distances. This degeneracy for the N-vortex problem enables to uncouple the computation of the good values of Γ_2/Γ_1 and the computation of the absolute equilibria.

VI Two polygons and one particle

In this section, we study the co-rotating points associated to a relative equilibrium formed of two concentric polygons and the particular case of an absolute equilibrium.

Theorem 6.1 *Let us consider a relative equilibrium formed of two concentric regular n-gons with the same vorticities on each n-gon. Let us also assume that one of the two following assumptions is true:*

- *Γ_1 and Γ_2 have the same sign and s_2/s_1 has argument $2K\pi/n$.*
- *Γ_1 and Γ_2 have opposite signs and s_2/s_1 has argument $2(K+1/2)\pi/n$.*

Let us consider the configuration formed of the two polygons and a vortex with zero vorticity at position z. If the configuration is a relative equilibrium, then z/s_1 has argument $K'\pi/n$.

Proof. Let v be the velocity of the vortex with zero velocity and ω be the angular velocity of the configuration formed of the two polygons. If this configuration remains a relative equilibrium when we also consider the particle with zero vorticity, we necessarily have:

$$v = \frac{1}{n}((v - v_{0,1}) + \ldots + (v - v_{n-1,1})) + \frac{1}{n}(v_{0,1} + \ldots + v_{n-1,1})$$
$$= \frac{i\omega}{n}((z - z_{0,1}) + \ldots + (z - z_{n-1,1})) + \frac{i\omega}{n}(z_{0,1} + \ldots + z_{n-1,1}) = i\omega z,$$

according to Proposition 2.3. According to Proposition 2.2, this relation is equivalent to:

$$\frac{in\Gamma_1\bar{z}^{n-1}}{\bar{z}^n-\bar{s}_1^n}+\frac{in\Gamma_2\bar{z}^{n-1}}{\bar{z}^n-\bar{s}_2^n}=i\omega z\cdot$$

It implies:

$$\frac{\Gamma_1|z|^{2n}}{|z^n-s_1^n|^2}-\frac{\Gamma_1\bar{z}^n s_1^n}{|z^n-s_1^n|^2}+\frac{\Gamma_2|z|^{2n}}{|z^n-s_2^n|^2}-\frac{\Gamma_2\bar{z}^n s_2^n}{|z^n-s_2^n|^2}=\bar{z}^n\left(\frac{\Gamma_1(z^n-s_1^n)}{|z^n-s_1^n|^2}+\frac{\Gamma_2(z^n-s_2^n)}{|z^n-s_2^n|^2}\right)$$

$$=\bar{z}^n\left(\frac{\Gamma_1}{\bar{z}^n-\bar{s}_1^n}+\frac{\Gamma_2}{\bar{z}^n-\bar{s}_2^n}\right)=\frac{\omega|z|^2}{n}\in\mathbb{R}\cdot$$

If $z\neq 0$, this implies:

$$\left(\frac{\Gamma_1}{|z^n-s_1^n|^2}+\frac{\Gamma_2\left(\frac{s_2}{s_1}\right)^n}{|z^n-s_2^n|^2}\right)\left(\frac{s_1}{z}\right)^n=\frac{\Gamma_1\left(\frac{s_1}{z}\right)^n}{|z^n-s_1^n|^2}+\frac{\Gamma_2\left(\frac{s_2}{z}\right)^n}{|z^n-s_2^n|^2}$$

$$=\frac{1}{|z|^{2n}}\left(\frac{\Gamma_1\bar{z}^n s_1^n}{|z^n-s_1^n|^2}+\frac{\Gamma_2\bar{z}^n s_2^n}{|z^n-s_2^n|^2}\right)\in\mathbb{R}\cdot$$

If Γ_1 and Γ_2 have the same sign and s_2/s_1 has argument $2K\pi/n$ or if Γ_1 and Γ_2 have opposite signs and s_2/s_1 has argument $2(K+1/2)\pi/n$, the first factor cannot vanish and we have: $(s_1/z)^n\in\mathbb{R}$. QED.

Theorem 6.2 *Let us consider the configuration formed of the absolute equilibrium computed in Theorem 5.2 and a vortex with zero vorticity at position z. If $z=0$, this configuration is an absolute equilibrium. And there exists exactly one absolute equilibrium with $z\neq 0$, for which z belongs to a ray containing 0 and a vortex of the internal polygon.*

Proof. Let us suppose, for instance: $|\Gamma_2|>|\Gamma_1|$. According to Theorem 5.2, this implies that the first polygon is the internal polygon. If $z\neq 0$, according to Proposition 2.2, the configuration is an absolute equilibrium if, and only if:

$$\frac{\Gamma_1}{1-\left(\frac{\bar{s}_1}{\bar{z}}\right)^n}=-\frac{\Gamma_2}{1-\left(\frac{\bar{s}_2}{\bar{z}}\right)^n}\cdot$$

According to the proof of Theorem 5.2, this is equivalent to:

$$\frac{\Gamma_1}{1-\left(\frac{\bar{s}_1}{\bar{z}}\right)^n}=-\frac{\Gamma_2}{1+\left(\frac{\Gamma_2}{\Gamma_1}\right)^2\left(\frac{\bar{s}_1}{\bar{z}}\right)^n},$$

$$\left(\frac{z}{s_1}\right)^n=\frac{\Gamma_2}{\Gamma_1}\times\frac{1-\frac{\Gamma_2}{\Gamma_1}}{1+\frac{\Gamma_2}{\Gamma_1}}=-\frac{|\Gamma_2|}{|\Gamma_1|}\times\frac{1+\frac{|\Gamma_2|}{|\Gamma_1|}}{1-\frac{|\Gamma_2|}{|\Gamma_1|}}>0$$

as $\Gamma_2/\Gamma_1<0$ (according to Theorem 5.2) and $|\Gamma_2|>|\Gamma_1|$. QED.

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